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Equivalence in Two-, Three-, and Four-Dimensional Space-Times

Ø. Grøn[1,2] and E. Eriksen[1]

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In a recent discussion E. A. Desloge has performed an analysis of the kinematic nonequivalence of a uniformly accelerating reference frame (UAR) and a frame at rest in a uniform gravitational field (UGF). Desloge considered two-dimensional space-times. We here review the geometrical description of gravity in two, three, and four space-time dimensions in order to discuss the *dynamical* nonequivalence of UAR and UGF both in two and in four space-time dimensions. We also consider the motion of photons in UAR and UGF in order to illustrate some relativistic effects of a kinematic nature.

1. INTRODUCTION

Recently there has been a significant amount of research on the properties of space-times of two dimensions (Brown, 1988; Brown *et al.*, 1986; Calheiros and Maia, 1988; Katanaev and Volovich, 1990; Sanches, 1986; Jackiw, 1985; Gegenberg *et al.*, 1988; Mann *et al.*, 1990, 1991; Sikkema and Mann, 1991; Kelly and Mann, 1991; Mann, 1991; Xu and Zhu, 1991; Schmidt, 1991; Mohammedi, 1990) and three dimensions (Staruszkiewicz, 1963; Collas, 1977; Clement, 1976, 1985, 1990; Gott and Alpert, 1984; Gott *et al.*, 1986; Giddings *et al.*, 1984; Deser *et al.*, 1984; Deser and Jackiw, 1984; Deser and Mazur, 1985; Deser, 1985; Deser and Laurent, 1986; Deser and Jackiw, 1989; Jackiw, 1990; Deser, 1990; Barrow *et al.*, 1986; Melvin, 1986; Dereli and Tucker, 1988; Hall *et al.*, 1987; Burges, 1985; Bezerra, 1987; Vurio, 1985; Percacci *et al.*, 1987; Brown and Henneaux, 1986; Nutku and Baekler, 1989; Grignani and Lee, 1989; Ortiz, 1990*a*,-*c*; Menotti and Seminara, 1991; Ishida *et al.*, 1990; Linet, 1990; Fujiwara and Soda, 1990;

[1]Department of Physics, University of Oslo, Box 1048 Blindern, N-0316 Oslo 3, Norway.
[2]Oslo College of Engineering, N-0254 Oslo 2, Norway.

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Soda, 1990; Ward, 1990; Keszthelyi *et al.*, 1991; Cho *et al.*, 1991; Williams, 1991; Waelbroeck, 1991; Cornish and Frankel, 1991; Duncan and Ihrig, 1991).

There are several reasons for this interest in lower-dimensional gravity. Jackiw (1985) has stated that his principal reason for studying such models is pedagogical, and proceeds: "Just as lower-dimensional, nongravitational field theories are used for studying effects relevant to our world, (e.g. spontaneous symmetry breaking, anomalies, confinement, solitons, phase transitions, and tunnelling etc.), so also I hope that lower dimensional gravity can illuminate the physical four (and possibly higher) dimensional models." Furthermore, he notes that three-dimensional field theories provide a phenomenological description of four-dimensional physics at high temperature.

Also, it has turned out that two- and three-dimensional theories of gravity have provided a deeper insight into the theory of strings. For example, the solutions of the field equations for point masses in $(2+1)$-dimensional space-time lead directly to the solutions for strings in $(3+1)$-dimensional space-time (Barrow *et al.*, 1986; Gott, 1985).

In a recent article Desloge (1989) has investigated the "nonequivalence of a uniformly accelerating reference frame and a frame at rest in a uniform gravitational field" by considering two-dimensional space-times. His discussion and conclusions are interesting. These two-dimensional space-times have independently been investigated by Tagaki (1989).

We shall here discuss how far Desloge's work can be taken over to four space-time dimensions. In this connection it is important to make the distinction between relativistic kinematics and relativistic dynamics. In relativistic kinematics the line element is given and one investigates the motion of free particles as given by the timelike or lightlike geodesic curves in space-time. In relativistic dynamics one considers the effects of matter on the curvature of space-time, as given by Einstein's field equations.

Desloge performed a kinematic analysis of a two-dimensional space-time. We shall extend his investigation and make a dynamical investigation of the space-time he considered, and of the corresponding four-dimensional space-time.

2. TWO-DIMENSIONAL SPACE-TIMES

According to the general theory of relativity the space-time geometry is obtained from the field equations

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu}, \qquad \kappa = 8\pi G/c^4 \tag{1}$$

Here Λ is the cosmological constant, $g_{\mu\nu}$ the metric tensor, $T_{\mu\nu}$ the energy-momentum density tensor, and $G_{\mu\nu}$ the Einstein curvature tensor

$$G_{\mu\nu} = R_{\mu\nu} - (1/2)Rg_{\mu\nu} \tag{2}$$

$R_{\mu\nu}$ is the Ricci tensor and $R = R^{\mu}{}_{\mu}$ the Ricci curvature scalar. The Ricci tensor is the contracted tensor $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$. The Riemann curvature tensor $R_{\mu\nu\alpha\beta}$ contains the complete information about the geometry of space-time.

In two-dimensional space-times there is only one independent component of the Riemann curvature tensor, say R_{1212}. It represents the Gaussian curvature K of the two-dimensional space-time (Weinberg, 1972)

$$K = -R_{1212}/g \tag{3}$$

where g is the determinant of the metrical tensor. The Ricci tensor and the Ricci curvature scalar can now be expressed as

$$R_{\mu\nu} = -Kg_{\mu\nu}, \qquad R = -2K \tag{4}$$

It follows that the Einstein curvature tensor vanishes for every two-dimensional space-time.

Thus, every two-dimensional space-time with $\Lambda = 0$ must have vanishing energy-momentum tensor, and there is no dynamical theory of space-time. As stated by Jackiw (1985), "Gravity on a line must be invented anew."

The commonly adopted equation for two-dimensional gravity is the constant curvature equation

$$R + b\Lambda = 0, \qquad b = \text{const} \tag{5}$$

Desloge (1989) considered the line element

$$ds^2 = \alpha^2(x)c^2\,dt^2 - dx^2 \tag{6}$$

In this case the field equation (5) takes the form

$$\alpha'' - 2b\Lambda\alpha = 0 \tag{7}$$

The general solution is

$$\alpha = C_1\,e^{Hx} + C_2\,e^{-Hx}, \qquad H = (2b\Lambda)^{1/2}, \qquad b\Lambda > 0 \tag{8}$$

$$\alpha = C_1\cos(H_1x) + C_2\sin(H_1x), \qquad H_1 = (-2b\Lambda)^{1/2}, \qquad b\Lambda < 0 \tag{9}$$

$$\alpha = C_1 + C_2x, \qquad \Lambda = 0 \tag{10}$$

Desloge discussed two particular line elements of the form (6), namely

$$ds^2 = e^{2gx/c^2}c^2\,dt^2 - dx^2 \tag{11}$$

and

$$ds^2 = \left(1 + \frac{gx}{c^2}\right)^2 c^2\,dt^2 - dx^2 \tag{12}$$

Using the geodesic equation, Desloge showed that the acceleration of a free particle instantaneously at rest in a space-time described by the line element (11) is the same, equal to g, at all points in the field. This line element represents a rigid frame in a uniform gravitational field. The line element (12), on the other hand, represents a uniformly accelerated rigid frame in flat space-time. Desloge used these line elements to discuss to what extent observations made in a rigid enclosure at rest in a gravitational field are not equivalent to observations made in a rigid enclosure that is uniformly accelerated in field free space.

The line element (11) is obtained from (8) by choosing $C_1 = 1$, $C_2 = 0$, $b = 2g^2/c^4\Lambda$, which according to equation (5) gives $R = -2g^2/c^4$. The line element (12) is obtained by putting $C_1 = 1$, $C_2 = g/c^2$ in equation (10).

Some interesting properties of the space-time described by the line element (11) should be noted. Takagi (1989) has shown that this space-time is geodesically incomplete. He found its maximal analytic extension and noted its relation to the anti-de Sitter space-time.

Mann (1991) has made a detailed study of two-dimensional space-times described as solutions of the field equations (5) (generalized by including a source-term proportional to the trace of the energy-momentum tensor). In particular he constructed a solution described by (8) in a region $|x| < \alpha_0$ and by (10) for $|x| \geq \alpha_0$, where α_0 is a boundary position. He showed that this solution represents an $(1+1)$-dimensional analog of the $(3+1)$-dimensional false vacuum bubble solution. He also discussed two-dimensional black-hole solutions.

Let us summarize our results so far. In two-dimensional space-time, the Einstein curvature tensor vanishes identically. Thus, the general theory of relativity has no dynamical equations as applied to two-dimensional space-time. Recently, however, researchers on two-dimensional space-times have used a theory based upon the field equation (5). If this is applied to the particular line elements (11) and (12) considered by Desloge, one finds that they correspond to two different situations. Equation (11) represents space-time with constant, negative curvature and a nonvanishing cosmological constant, while (12) represents Minkowski space-time with reference to a uniformly accelerated frame.

3. THREE-DIMENSIONAL SPACE-TIMES

We are now going to consider "gravity on a surface" according to general relativity. In this case space-time is three-dimensional, and the Riemann curvature tensor can be expressed by the Einstein tensor as follows (Jackiw, 1985)

$$R_{\alpha\beta\mu\nu} = -\varepsilon_{\alpha\beta\gamma}\varepsilon_{\mu\nu\delta}G^{\gamma\delta} \tag{13}$$

where $\varepsilon_{\alpha\beta\gamma}$ is the completely antisymmetric tensor. In the absence of matter, and with vanishing cosmological constant, the Einstein tensor vanishes. Then $R_{\alpha\beta\mu\nu}=0$. Thus, every three-dimensional vacuum space-time with $\Lambda=0$ is flat.

In the case of a vacuum with $\Lambda\neq 0$, the field equations can be written (Sanches, 1986)

$$R_{\alpha\beta\mu\nu}=\Lambda(g_{\alpha\mu}g_{\beta\nu}-g_{\alpha\nu}g_{\mu\beta}) \tag{14}$$

All spaces that are solutions of (14) are of constant curvature. For $\Lambda>0$ one gets a three-dimensional version of the de Sitter space-time (Jackiw, 1985).

Even if the vacuum field equations with $\Lambda=0$ outside a particle are solved only by flat space-time, there are interesting nonlocal properties of this solution. The three-dimensional space-time outside a particle was first described by Staruszkiewicz (1963) and has later been thoroughly examined in several works (Clement, 1976, 1985, 1990; Gott and Alpert, 1984; Gott *et al.*, 1986; Giddings *et al.*, 1984; Deser *et al.*, 1984; Deser and Jackiw, 1989).

Space-time outside a particle of mass M is described by the line element

$$ds^2=c^2\,dt^2-\left(1-\frac{\kappa}{2\pi}M\right)^{-2}dr^2-r^2\,d\Phi^2 \tag{15}$$

The transformation

$$r'=\left(1-\frac{\kappa}{2\pi}M\right)^{-1}r,\qquad \phi'=\left(1-\frac{\kappa}{2\pi}M\right)\phi \tag{16}$$

leads to

$$ds^2=c^2\,dt^2-dr'^2-r'^2\,d\phi^2 \tag{17}$$

which represents flat space-time. But the limits of ϕ' are different from the limits of ϕ: $0\leq\phi\leq 2\pi$ and $0\leq\phi'\leq 2\pi-\kappa M$. The line element (17) thus describes a cone with angle deficit κM at its induced by the presence of mass.

The four-dimensional space-time outside a right, infinitely long, cosmic string is quite similar, with an angle deficit proportional to the mass per unit length of the string (Francisco and Matsas, 1989).

Even if space-time is flat, light passing the string is deflected with an angle equal to the angle deficit of the cone.

Three-dimensional space-times generated by circular and straight strings with and without tension have been thoroughly investigated by Deser and Jackiw (1989). Furthermore, it was noted by Barrow (1986) that by reduction of the four-dimensional domain-wall solution of Vilenkin (1983) one obtains a $(2+1)$-dimensional line source with metric

$$ds^2=e^{-k|x|}(dt^2-dx^2-e^{kt}\,dy^2)\qquad k=\text{const} \tag{18}$$

Since three-dimensional space-time is flat in vacuum between point masses, there is no gravitational attraction between particles. This also means that the three-dimensional version of general relativity has no Newtonian limit, since in Newtonian theory particles attract each other in two space dimensions (Clement, 1985; Barrow *et al.*, 1986).

The properties of a space-time described by the line element (6) are similar in three- and four-dimensional space-times, and will be discussed in the next section.

4. FOUR-DIMENSIONAL SPACE-TIME

We shall now investigate the four-dimensional generalization of the line element (6) that was studied by Desloge (1989),

$$ds^2 = \alpha^2(x)c^2\,dt^2 - dx^2 - dy^2 - dz^2 \tag{19}$$

The only nonvanishing components of Einstein's curvature tensor are

$$G_{yy} = G_{zz} = -\alpha''/\alpha \tag{20}$$

The field equations (1) then take the form

$$T_{00} = (\Lambda/\kappa)\alpha^2, \qquad T_{11} = -\Lambda/\kappa \tag{21}$$

$$T_{22} = T_{33} = (c^4/8\pi G)(\alpha''/\alpha - \Lambda) \tag{22}$$

In the case of a vanishing cosmological constant the vacuum equations reduce to $\alpha'' = 0$. This case represents flat space-time. The solution is given in equation (10). $C_2 = 0$ gives the Minkowski metric in an inertial reference frame, and $C_2 \neq 0$ gives the metric in a uniformly accelerating reference frame, which may be written

$$ds^2 = \left(1 + \frac{gx}{c^2}\right)^2 c^2\,dt^2 - dx^2 - dy^2 - dz^2 \tag{23}$$

Let us consider the solution corresponding to the line element (11),

$$ds^2 = e^{2gx/c^2}c^2\,dt^2 - dx^2 - dy^2 - dz^2 \tag{24}$$

Substituting this into equations (21) and (22) leads to

$$T^0_0 = T^1_1 = (c^4/8\pi G)\Lambda, \qquad T^2_2 = T^3_3 = (1/8\pi G)(c^4\Lambda - g^2) \tag{25}$$

Neither with $\Lambda = 0$ nor with $\Lambda \neq 0$ does the line element (19) represent a vacuum solution of Einstein's field equations. If $\Lambda = 0$, equation (25) gives

$$T^0_0 = T^1_1 = 0, \qquad T^2_2 = T^3_3 = -g^2/8\pi G \tag{26}$$

This represents a medium with vanishing rest mass density and with stresses parallel to the symmetry plane. The properties of this medium are not physically acceptable. We now consider the case $\Lambda > 0$.

Depending upon the value of Λ, there are several types of sources for the solution (24). We shall now consider two cases.

A. If $\Lambda = g^2/c^4$, equation (25) gives

$$T_0^0 = T_1^1 = g^2/8\pi G, \qquad T_2^2 = T_3^3 = 0 \tag{27}$$

This is an anisotropic medium with vanishing stresses along the yz plane and maximal compression in the x direction. The pressure p in this direction and the density ρ of the medium are connected by

$$p = \rho c^2 \tag{28}$$

An isotropic medium of this type is called a Zel'dovich fluid. In such a fluid the sound velocity is equal to the velocity of light c.

B. The trace of the energy-momentum tensor is

$$T = T_0^0 + T_1^1 + T_2^2 + T_3^3 = (1/4\pi G)(2c^4\Lambda - g^2) \tag{29}$$

If $\Lambda = g^2/2c^4$, the trace vanishes and

$$T_\nu^\mu = (g^2/16\pi G)\,\mathrm{diag}(1, 1, -1, -1) \tag{30}$$

This energy-momentum tensor represents a parallel electrostatic field (McWittie, 1929). According to this interpretation the line element, (24) represents a plane-symmetric universe with nonvanishing cosmological constant filled with an electrostatic field. An interpretation with the electrostatic field replaced by a magnetostatic field is also possible.

We have seen that the line element (19) does not permit a vacuum space-time except when $\alpha'' = 0$. The simplest static, plane-symmetric generalization of this line element is

$$ds^2 = \alpha^2(x)c^2\,dt^2 - dx^2 - \beta^2(x)(dy^2 + dz^2) \tag{31}$$

Solving the vacuum field equations with $\Lambda = 0$ for this line element, one finds

$$ds^2 = \left(1 - \frac{3gx}{c^2}\right)^{-2/3} c^2\,dt^2 - dx^2 - \left(1 - \frac{3gx}{c^2}\right)^{4/3}(dy^2 + dz^2) \tag{32}$$

The physical interpretation of this solution has been given by Amundsen and Grøn (1983). It describes the parallel gravitational field outside a massive plane of infinite extension.

5. KINEMATIC PROPERTIES OF THE SPACE-TIMES

The space-time of the uniformly accelerated reference frame described by the line element (23) is flat. There is a metric singularity at $x = -c^2/g$, where $g_{tt} = 0$. Light emitted from a source at this position toward $x = 0$ is infinitely red-shifted, and no information can pass from an event at $x < -c^2/g$ through this plane. It is therefore called an event horizon. Because space-time is flat, this event horizon is a property of the reference frame, which may be transformed away by going into, for example, an inertial frame. The horizon is not an invariant property of space-time. The space-time described by the line element (24) is curved, but free of any horizon. The space-time curvature is due to the nonvanishing cosmological constant and the electrical field energy.

The line element (32) represents a vacuum solution of Einstein's field equations with $\Lambda = 0$. It describes a curved space-time. The curvature is due to the existence of a massive plane at $x = 0$. The curvature diverges at a coordinate distance $x_0 = c^2/3g$ from the massive plane. Since $g_{xx} = -1$, this is just the physical distance to the curvature singularity as measured by standard measuring rods at rest in the x direction. The length of coordinate unit rods parallel to the yz plane shrinks by a factor $(1-3gx/c^2)^{2/3}$ with increasing distance from $x = 0$. Thus, the geometry of the xyz space may be compared to that of a birdcage of height $c^2/3g$.

As pointed out by Desloge (1989), the gravitational field of acceleration, i.e., the acceleration of a free particle instantaneously at rest as measured with standard clocks at rest in the reference frame, is uniform in the space-time (24), but not in (23). As an illustration of the different kinematic properties of these space-times, we shall now calculate the paths of photons emitted from $x = a$, $y = b$, $z = 0$ in the y direction.

We make use of the fact that Fermat's principle is valid in a static space-time (Misner *et al.*, 1973). This means that in the present case light (a photon) follows a path such that the coordinate travel time t is extremal.

Consider a light ray in the xy plane. The space-time interval between events on the curve vanishes. With the line element (19) this gives

$$\alpha^2(x)c^2\,dt^2 - dx^2 - dy^2 = 0 \tag{33}$$

Thus, the coordinate travel time is given by

$$ct = \int \alpha^{-1}(x)(dx^2 + dy)^{1/2} \tag{34}$$

or

$$ct = \int f\,dy \tag{35}$$

where

$$f[x, x'(y)] = \alpha^{-1}(x)[1 + x'(y)]^{1/2} \tag{36}$$

Since f does not depend explicitly upon the argument y, and since the integral (35) is extremal according to Fermat's principle, we may write down a quantity which is conserved along the path (Stephani, 1982)

$$x'(y)\frac{df}{dx'(y)}-f=\text{const} \tag{37}$$

This quantity corresponds to the Hamiltonian in classical mechanics (Goldstein, 1980).

In our case equation (37) leads to

$$\alpha^{-1}(x)[1+x'(y)^2]^{-1/2}=\alpha^{-1}(a) \tag{38}$$

which may be written

$$y'=\alpha(x)[\alpha^2(a)-\alpha^2(x)]^{-1/2} \tag{39}$$

This equation is easily integrated for the two metrics in question.

Inserting the metric (23), i.e., $\alpha=1+gx/c^2$, we get the result

$$(x+c^2/g)^2+(y-b)^2=(a+c^2/g)^2 \tag{40}$$

which describes a circle with center in $(-c^2/g, b)$ and radius $a+c^2/g$.

This path illustrates an interesting and nontrivial property about the kinematics of flat space-time as referred to a uniformly accelerated reference frame. Even though 3-space is Euclidean, a photon starting out with velocity c in the y direction ends up by moving into the horizon at $x=-c^2/g$ without any motion at all in the y direction. This is possible because of the gravitational time dilation in a gravitational field. The velocity of light is constant and equal to c *as measured locally*, but an observer, for example, at $x=0$ will measure a decreasing light velocity as the photon approaches the horizon, in accordance with redshift measurements, which shows to that observer that time goes slower far down in the gravitational field near the horizon.

Inserting the line element (24) of Desloges' uniform gravitational field, i.e., $\alpha=e^{gx/c^2}$, we get from equation (39) the solution

$$y-b=(c^2/g)\arccos e^{g(x-a)/c^2} \tag{41}$$

or

$$x-a=(c^2/g)\ln\cos[g(y-b)/c^2] \tag{42}$$

We see that all curves have the same shape (independent of a and b).

We also note that $y-b$ is restricted to the interval $\langle -\frac{1}{2}\pi c^2/g, \frac{1}{2}\pi c^2 g\rangle$. This means that two observers whose horizontal distance is larger than $\pi c^2/g$ cannot communicate directly by a light signal.

The curves given by equations (40) and (42) agree with those recently found by Desloge (1990).

6. CONCLUSION

The geometrical descriptions of gravity on a line and gravity on a surface have been reviewed. Some main points are as follows.

A. General relativity cannot describe two-dimensional space-times, because the Einstein curvature tensor is identically zero in this case. However, there exists a commonly accepted theory of gravity on a line. According to this theory, empty two-dimensional space-time has constant curvature.

B. According to Einstein's field equations, empty three-dimensional space-time with $\Lambda = 0$ is flat. Point particles do not attract each other, and there is no Newtonian limit.

In general there is great difference between gravity on a line, on a plane, and in space. Desloge investigated kinematically the nonequivalence of a uniformly accelerating reference frame and a frame at rest in a uniform gravitational field, by studying a two-dimensional line element. He obtained some interesting results. One would like, therefore, to know the dynamical character of the space-time upon which he based his kinematic analysis and of its four-dimensional extension.

We have investigated this question in the present work. Our results can be summarized as follows.

C. The two-dimensional uniform gravitational field of Desloge, given by equation (11), cannot be extended to four-dimensional space-time without introducing a nonvanishing energy-momentum density tensor.

D. The four-dimensional extension of Desloge's line element can be interpreted to describe a plane-symmetric universe with $\Lambda > 0$ and an electrostatic or a magnetostatic field.

E. Einstein's field equations do not permit the existence of empty space-time with a uniform gravitational field, i.e., a field in which the proper acceleration of a free particle instantaneously at rest is the same everywhere in the field.

F. The closest one can come to a uniform gravitational field in *empty* four-dimensional space-time is the parallel gravitational field outside a massive plane of infinite extension, described by the line element (32).

Finally, part of Desloge's illustration of the different kinematic properties of space-time in a uniformly accelerated reference frame and in a uniform gravitational field has been worked out in a new way, confirming the validity of Desloge's results.

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